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On some spectra of shifts

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Abstract

The spectrum, the left (right) spectrum, the approximate point (defect) spectrum and the point spectrum of forward (backward) unilateral shift on c_0, c, ℓ_∞, ℓ_p, as well as of forward (backward) bilateral shift on $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$, $p \geq 1$, are determined.

1 Introduction

Let $\mathbb{C}$ be the set of all complex numbers and let X be an infinite dimensional complex Banach space. We use $B(X)$ to denote the set of all bounded linear operators on X. This is a Banach algebra. Let I denote the identity operator. The group of all invertible operators is denoted by $B(X)^{-1}$, while the semigroups of left and right invertible operators are denoted by $B(X)_l^{-1}$ and $B(X)_r^{-1}$, respectively. For $A \in B(X)$ we use $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, to denote the null-space and the range of A.

The spectrum of $A \in B(X)$ is

$$\sigma(A) = \sigma_l(A) \cup \sigma_r(A), \tag{1.1}$$

where

$$\begin{aligned} \sigma_l(A) &= \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not left invertible}\}, \\ \sigma_r(A) &= \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not right invertible}\} \end{aligned}$$

are the left and the right spectrum of A, respectively.

For $\lambda \in \mathbb{C}$ and $A \in B(X)$ the following implication holds [3, Posledica 5.3.3]:

$$|\lambda| > \|A\| \Longrightarrow A - \lambda I \in B(X)^{-1}, \tag{1.2}$$

and hence,

$$\sigma(A) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|A\|\}. \tag{1.3}$$

The point spectrum of A is defined by

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not injective}\}.$$

The *injectivity modulus* (*minimum modulus*) of $A \in B(X)$ is defined as

$$j(A) = \inf_{\|x\|=1} \|Ax\|.$$

We immediately obtain that $\|Ax\| \geq j(A)\|x\|$ for every $x \in X$, and

$$j(A) = \max\{c \geq 0 \ : \ \|Ax\| \geq c\|x\|, \quad \text{for every } x \in X\}.$$

An operator $A \in B(X)$ is *bounded below* if there exists some $c > 0$ such that

$$c\|x\| \leq \|Ax\|, \text{ for every } x \in X.$$

It is easy to see that A is bounded below if and only if $j(A) > 0$.

Recall that $A \in B(X)$ is bounded below if and only if A is injective and $\mathcal{R}(A)$ is closed (see [3, Teorema 4.7.2]).

The approximate point spectrum of A is defined by

$$\sigma_a(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not bounded below}\},$$

while the approximate defect spectrum of A is defined by

$$\sigma_\delta(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not surjective}\}.$$

For $K \subset \mathbb{C}$, ∂K denotes the boundary of K.

For $A \in B(X)$ it is well-known that (see [3, p. 186, 187, Teorema 5.9.13, Posledica 5.9.18])

$$\sigma_p(A) \subset \sigma_a(A) \subset \sigma_l(A), \tag{1.4}$$

$$\sigma_\delta(A) \subset \sigma_r(A), \tag{1.5}$$

as well as,

$$\partial\sigma(A) \quad \subset \quad \sigma_a(A) \cap \sigma_\delta(A) \subset \sigma_l(A) \cap \sigma_r(A). \tag{1.6}$$

The spectra $\sigma_a(A)$, $\sigma_\delta(A)$, $\sigma_l(A)$, $\sigma_r(A)$ are always closed and non-empty, while the point spectrum $\sigma_p(A)$ can be non-closed and empty.

If $A \in B(X)^{-1}$, then [3, Teorema 5.4.12]

$$\sigma(A^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(A)\}. \tag{1.7}$$

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $\mathbb{C}^{\mathbb{N}_0}$ be the linear space of all complex sequences $x = (x_k)_{k=0}^{+\infty}$. Let ℓ_∞, c and c_0 denote the set of bounded, convergent and null sequences. We write $\ell_p = \{x \in \mathbb{C}^{\mathbb{N}_0} : \sum_{k=0}^{+\infty} |x_k|^p < +\infty\}$ for $1 \leq p < \infty$. For $n = 0, 1, 2, \ldots$, let $e^{(n)}$ denote the sequences such that $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. The forward and the backward unilateral shifts U and V are linear operators on $\mathbb{C}^{\mathbb{N}_0}$ defined by

$$Ue^{(n)} = e^{(n+1)} \quad \text{and} \quad Ve^{(n+1)} = e^{(n)}, \quad n = 0, 1, 2, \ldots .$$

Invariant subspaces for U and V include c_0, c, ℓ_∞ and ℓ_p, $p \geq 1$. Recall that for every $1 \leq p < \infty$,

$$\ell_1 \subset \ell_p \subset c_0 \subset c \subset \ell_\infty, \tag{1.8}$$

and for each $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $U,\ V \in B(X)$ and $\|U\| = \|V\| = 1$. On the Hilbert space ℓ_2 we also have that $V = U^*$, where U^* is the Hilbert-adjoint of U.

2 Spectra

We shall write $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\mathbb{S} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Theorem 2.1. *For each $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $p \geq 1$, and the forward and backward unilateral shifts U, $V \in B(X)$ there are equalities*

$$\sigma_\delta(U) = \sigma_r(U) = \sigma(U) = \mathbb{D}, \tag{2.1}$$

$$\sigma_a(V) = \sigma_l(V) = \sigma(V) = \mathbb{D}. \tag{2.2}$$

Proof. From

$$\|U\| = \|V\| = 1, \tag{2.3}$$

according to (1.3), it follows that

$$\sigma(U) \cup \sigma(V) \subset \mathbb{D}. \tag{2.4}$$

Observe that $VU = I$,

$$V(I - UV) = 0 \neq I - UV,$$

and also

$$\mathcal{N}(V) = (I - UV)X \neq \{0\}. \tag{2.5}$$

From (2.3) and (1.2) it is clear that

$$|\lambda| < 1 \Longrightarrow I - \lambda U \in B(X)^{-1}. \tag{2.6}$$

For $|\lambda| < 1$, since $V - \lambda = V(I - \lambda U)$, from (2.5) and (2.6) it follows

$$\mathcal{N}(V - \lambda I) = (V - \lambda)^{-1}(0) = (I - \lambda U)^{-1} V^{-1}(0) \neq \{0\}. \tag{2.7}$$

This means that $V - \lambda I$ is not injective for $\lambda \in \mathbb{C}$, $|\lambda| < 1$, and hence,

$$\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_a(V) \subset \sigma_l(V) \subset \sigma(V). \tag{2.8}$$

Since $\sigma_a(V)$ is closed, from (2.8) we obtain

$$\mathbb{D} \subset \sigma_a(V) \subset \sigma_l(V) \subset \sigma(V). \tag{2.9}$$

It is obvious that $e^{(0)} \notin \mathcal{R}(U)$. Suppose that $\lambda \in \mathbb{C}$ and $0 < |\lambda| < 1$. We show that $e^{(0)} \notin \mathcal{R}(\lambda I - U)$. If there exists $x = (x_k)_{k=0}^{+\infty}$ such that $(\lambda I - U)x = e^{(0)}$, then

$$(\lambda x_0, \lambda x_1 - x_0, \lambda x_2 - x_1, \dots) = (1, 0, 0, \dots)$$

and hence

$$x = (\frac{1}{\lambda}, \frac{1}{\lambda^2}, \frac{1}{\lambda^3}, \dots),$$

which is not a bounded sequence and so, it is not in X.

Therefore, $\lambda I - U$ is not surjective for every $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$ and hence,

$$\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_\delta(U) \subset \sigma_r(U) \subset \sigma(U).$$

As $\sigma_\delta(U)$ is closed, from the last inclusion we get

$$\mathbb{D} \subset \sigma_\delta(U) \subset \sigma_r(U) \subset \sigma(U). \tag{2.10}$$

From (2.4), (2.9) and (2.10) it follows (2.1) and (2.2). □

Theorem 2.2. *For each $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $p \geq 1$, and the forward and backward unilateral shifts U, $V \in B(X)$ there are equalities*

$$\sigma_a(U) = \sigma_l(U) = \mathbb{S}, \tag{2.11}$$

and

$$\sigma_\delta(V) = \sigma_r(V) = \mathbb{S}. \tag{2.12}$$

Proof. For $\lambda \neq 0$, $|\lambda| < 1$, since $|\lambda|^{-1} > 1 = \|V\|$ it follows that $V - \lambda^{-1}I \in B(X)^{-1}$ by (1.2). As $VU = I$, we have

$$V(\lambda I - U) = \lambda V - I = \lambda(V - \lambda^{-1}I),$$

and consequently, $\lambda I - U \in B(X)_l^{-1}$. Since also $U \in B(X)_l^{-1}$, for every $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$ we have that $\lambda \notin \sigma_l(U)$ and since $\sigma_l(U) \subset \sigma(U) = \mathbb{D}$ (Theorem 2.1), we conclude that

$$\sigma_l(U) \subset \mathbb{S}. \tag{2.13}$$

From (2.1), (1.6) and (1.4) we get

$$\mathbb{S} = \partial\sigma(U) \subset \sigma_a(U) \subset \sigma_l(U). \tag{2.14}$$

Now, from (2.13) and (2.14) we obtain that $\sigma_a(U) = \sigma_l(U) = \mathbb{S}$.

For $\lambda \neq 0$, $|\lambda| < 1$, since $|\lambda|^{-1} > 1 = \|U\|$, from (1.2) it follows that $\lambda^{-1}I - U \in B(X)^{-1}$ and since $VU = I$, from

$$V - \lambda I = V - \lambda VU = \lambda V(\lambda^{-1}I - U)$$

we conclude that $V - \lambda I$ is right invertible as a product of one invertible and one right invertible operator. As also $V \in B(X)_r^{-1}$, we have that $\lambda \notin \sigma_r(V)$ for every $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$ and since $\sigma_r(V) \subset \sigma(V) = \mathbb{D}$, we conclude that

$$\sigma_r(V) \subset \mathbb{S}. \tag{2.15}$$

From (2.2) we have that $\partial\sigma(V) = \mathbb{S}$ and from (1.6) and (1.5) it follows that

$$\mathbb{S} \subset \sigma_\delta(V) \subset \sigma_r(V). \tag{2.16}$$

Now, (2.15) and (2.16) imply (2.12). □

Remark that $\sigma_p(U) = \emptyset$. Indeed, U is injective and for $\lambda \neq 0$, from $(U - \lambda I)x = 0$ for $x = (x_0, x_1, x_2, \dots)$ it follows that $(-\lambda x_0, x_0 - \lambda x_1, x_1 - \lambda x_2, \dots) = (0, 0, 0, \dots)$, that is

$$-\lambda x_0 = 0,\ x_0 - \lambda x_1 = 0,\ x_1 - \lambda x_2 = 0, \dots,$$

and hence, $x_0 = 0, x_1 = 0, x_2 = 0, \dots$, i.e. $x = 0$.

Let $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$. Consider the sequence

$$x_\lambda = (1, \lambda, \lambda^2, \dots, \lambda^n, \dots).$$

From $|\lambda|^p < 1$, for $1 \leq p < +\infty$, it follows that $\sum_{k=0}^{+\infty} |\lambda^k|^p = \sum_{k=0}^{+\infty} (|\lambda|^p)^k < +\infty$ and so, $x_\lambda \in \ell_p$. According to the inclusions (1.8) we have that $x_\lambda \in X$ for each $X \in \{\ell_p, c_0, c, \ell_\infty\}$, $p \geq 1$.

Since $x_\lambda \neq 0$ and

$$Vx_\lambda = (\lambda, \lambda^2, \dots, \lambda^n, \dots) = \lambda(1, \lambda, \lambda^2, \dots, \lambda^n, \dots) = \lambda x_\lambda,$$

we conclude that $V - \lambda I$ is not injective and hence, $\lambda \in \sigma_p(V)$. Therefore,

$$\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_p(V). \tag{2.17}$$

Let $\lambda \in \mathbb{C}$, $|\lambda| = 1$ and $(V - \lambda)x = 0$ for $x = (x_k)_{k=0}^{+\infty} \in X$, where $X \in \{\ell_p, c_0\},\ p \geq 1$. Then $(x_1, x_2, x_3, \dots) = (\lambda x_0, \lambda x_1, \lambda x_2, \dots)$ and so,

$$x_1 = \lambda x_0,\ x_2 = \lambda x_1,\ x_3 = \lambda x_2, \dots.$$

Therefore, for $n \in \mathbb{N}$ we have

$$x_1 = \lambda x_0,\ x_2 = \lambda x_1 = \lambda^2 x_0, \quad \dots \quad, x_n = \lambda^n x_0. \tag{2.18}$$

Since $\lim\limits_{n\to\infty} x_n = 0$, it follows that $\lim\limits_{n\to\infty} |x_n| = 0$. As $|\lambda| = 1$, from (2.18) it follows that $|x_n| = |x_0|$ for all $n \in \mathbb{N}$ and hence, $\lim\limits_{n\to\infty} |x_n| = |x_0|$. Therefore, $x_0 = 0$ and so, $x_i = 0$ for all $i \in \mathbb{N}$, i.e. $x = 0$. Consequently, $\lambda \notin \sigma_p(V)$, and hence

$$\mathbb{S} \cap \sigma_p(V) = \emptyset. \tag{2.19}$$

Since $\sigma_p(V) \subset \sigma(V) = \mathbb{D}$, from (2.17) and (2.19) it follows that

$$\sigma_p(V) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}.$$

We remark that $\sigma_p(V)$ is not a closed set.

Now we consider the backward shift V on c. For the sequence $e = (1, 1, \dots)$ we have that $e \in c$, $e \neq 0$, $Ve = e$, and so $1 \in \sigma_p(V)$. Let $\lambda \in \mathbb{C}$, $|\lambda| = 1$, $\lambda \neq 1$ and $(V - \lambda)x = 0$ for $x = (x_k)_{k=0}^{+\infty} \in c$. According to (2.18) we have that

$$x = (x_0, \lambda x_0, \lambda^2 x_0, \lambda^3 x_0, \dots) = x_0(1, \lambda, \lambda^2, \lambda^3, \dots) \in c. \tag{2.20}$$

Since $|\lambda| = 1$ and $\lambda \neq 1$ the sequence $(1, \lambda, \lambda^2, \lambda^3, \dots)$ does not converge and hence from (2.20) it follows that $x_0 = 0$. Consequently, $x = 0$ and we get that $V - \lambda$ is an injection for all $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $\lambda \neq 1$. Now from (2.17) we conclude that

$$\sigma_p(V) = \{\lambda \in \mathbb{C} : |\lambda| < 1\} \cup \{1\}.$$

We remark again that $\sigma_p(V)$ is not closed.

Consider the backward shift V on ℓ_∞. Let $\lambda \in \mathbb{C}$ and $|\lambda| = 1$. Then $x_\lambda = (1, \lambda, \lambda^2, \dots, \lambda^n, \dots) \in \ell_\infty$, $x_\lambda \neq 0$ and $Vx_\lambda = \lambda x_\lambda$ and hence, $\lambda \in \sigma_p(V)$. Therefore, $\mathbb{S} \subset \sigma_p(V)$, which together with (2.17) gives $\mathbb{D} \subset \sigma_p(V) \subset \sigma(V) = \mathbb{D}$, and so

$$\sigma_p(V) = \mathbb{D}.$$

Let $\mathbb{C}^{\mathbb{Z}}$ be the linear space of all complex sequences $x = (x_k)_{k=-\infty}^{+\infty}$. Let $c_0(\mathbb{Z})$ be the set of all sequences $x = (x_k)_{k=-\infty}^{+\infty}$ such that $\lim\limits_{k\to\infty} x_k = \lim\limits_{k\to\infty} x_{-k} = 0$, i.e. $x_k \to 0$ when $|k| \to \infty$. For $x = (x_k)_{k=-\infty}^{+\infty} \in c_0(\mathbb{Z})$ set $\|x\| = \sup\limits_k |x_k|$. We write $\ell_p(\mathbb{Z}) = \{x \in \mathbb{C}^{\mathbb{Z}} : \sum_{k=-\infty}^{+\infty} |x_k|^p < \infty\}$ for $1 \leq p < \infty$, and for $x = (x_k)_{k=-\infty}^{+\infty} \in \ell_p(\mathbb{Z})$, $\|x\| = \left(\sum_{k=-\infty}^{+\infty} |x_k|^p\right)^{1/p}$. Remark that $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$ are Banach spaces.

For $k = \dots, -2, -1, 0, 1, 2, \dots$, let $\delta^{(k)}$ denote the sequences such that $\delta_k^{(k)} = 1$ and $\delta_i^{(k)} = 0$ for $i \neq k$. The forward and the backward bilateral shifts W_1 and W_2 are linear operators on $\mathbb{C}^{\mathbb{Z}}$ defined by

$$W_1\delta^{(k)} = \delta^{(k+1)} \quad \text{and} \quad W_2\delta^{(k+1)} = \delta^{(k)}, \quad k = \dots, -2, -1, 0, 1, 2, \dots.$$

Obviously, $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$, $p \geq 1$ are invariant subspaces for W_1 and W_2, and $W_1^{-1} = W_2$. For each $X \in \{c_0(\mathbb{Z}), \ell_p(\mathbb{Z})\}$, W_1 and W_2 are isometries. On the Hilbert space $\ell_2(\mathbb{Z})$ we have that $W_2 = W_1^*$, that is W_1 and W_2 are unitary.

Theorem 2.3. *If X is one of $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$, $p \geq 1$, then for the forward and backward bilateral shifts $W_1,\ W_2 \in B(X)$ there are equalities*

$$\sigma(W_1) = \sigma(W_2) = \mathbb{S}. \tag{2.21}$$

Proof. Since

$$\|W_1\| = \|W_2\| = 1, \tag{2.22}$$

from (1.3) it follows that

$$\sigma(W_1) \cup \sigma(W_2) \subset \mathbb{D}. \tag{2.23}$$

Let $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < 1$. Then $1/|\lambda| > 1 = \|W_2\|$ and from (1.3) it follows that $1/\lambda \notin \sigma(W_2) = \sigma(W_1^{-1})$ and hence $\lambda \notin \sigma(W_1)$ according to (1.7). From (2.23) it follows that

$$\sigma(W_1) \subset \mathbb{S}.$$

Suppose that $\lambda \in \mathbb{C}$, $|\lambda| = 1$. We prove that $\mathcal{R}(\lambda I - W_1)$ does not contain $\delta^{(0)}$. For $x = (x_k)_{k=-\infty}^{+\infty} \in X$, from $(\lambda I - W_1)x = \delta^{(0)}$ we get

$$\ldots, \lambda x_{-2} - x_{-3} = 0, \lambda x_{-1} - x_{-2} = 0, \lambda x_0 - x_{-1} = 1, \lambda x_1 - x_0 = 0, \lambda x_2 - x_1 = 0, \ldots,$$

and hence

$$x_1 = \tfrac{1}{\lambda} x_0,\ x_2 = \tfrac{1}{\lambda^2} x_0,\ x_3 = \tfrac{1}{\lambda^3} x_0, \ldots$$
$$x_{-2} = \lambda x_{-1},\ x_{-3} = \lambda^2 x_{-1},\ \ldots\ .$$

From $\lim\limits_{k\to\infty} x_k = 0$, $\lim\limits_{k\to\infty} x_{-k} = 0$ and $|\lambda| = 1$ we conclude that $x_0 = 0$ and $x_{-1} = 0$ which contradict the fact that $\lambda x_0 - x_{-1} = 1$.

Consequently, $\sigma(W_1) = \mathbb{S}$, and hence also

$$\sigma(W_2) = \sigma(W_1^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(W_1)\} = \mathbb{S}.$$

□

From (2.23), (1.6), (1.4), (1.5) and (1.1) it follows that

$$\sigma_a(W_i) = \sigma_\delta(W_i) = \sigma_l(W_i) = \sigma_r(W_i) = \mathbb{S}, \quad i = 1, 2.$$

We shall prove that $\sigma_p(W_1) = \sigma_p(W_2) = \emptyset$.

Let $\lambda \in \mathbb{C}$, $|\lambda| = 1$ and $(\lambda I - W_1)x = 0$ for $x = (x_k)_{k=-\infty}^{+\infty} \in X$, $X \in \{c_0(\mathbb{Z}), \ell_p(\mathbb{Z})\}$, $p \geq 1$. Then

$$\dots, \lambda x_{-2} - x_{-3} = 0, \lambda x_{-1} - x_{-2} = 0, \lambda x_0 - x_{-1} = 0, \lambda x_1 - x_0 = 0, \lambda x_2 - x_1 = 0, \dots.$$

Therefore, for $n \in \mathbb{N}$ we have

$$x_1 = \lambda^{-1} x_0 = \overline{\lambda} x_0, \; x_2 = \overline{\lambda} x_1 = \overline{\lambda}^2 x_0, \quad \dots \quad, x_n = \overline{\lambda}^n x_0, \tag{2.24}$$

and

$$x_{-1} = \lambda x_0, \; x_{-2} = \lambda x_{-1} = \lambda^2 x_0, \quad \dots \quad, x_{-n} = \lambda^n x_0. \tag{2.25}$$

As $\lim\limits_{n \to \infty} x_n = 0$, we have $\lim\limits_{n \to \infty} |x_n| = 0$. From (2.24), since $|\overline{\lambda}| = 1$, it follows that $|x_n| = |x_0|$ for all $n \in \mathbb{N}$ and hence, $\lim\limits_{n \to \infty} |x_n| = |x_0|$. Therefore, $x_0 = 0$ (the same conclusion can be obtained from (2.25) and the fact that $\lim\limits_{n \to \infty} x_{-n} = 0$) and so $x_i = 0$ for all $i \in \mathbb{Z}$, i.e. $x = 0$. Consequently, $\lambda \notin \sigma_p(W_1)$, and hence $\mathbb{S} \cap \sigma_p(W_1) = \emptyset$. From (2.21) it follows that $\sigma_p(W_1) = \emptyset$.

The equality $\sigma_p(W_2) = \emptyset$ can be proved similarly.

Since the spectrum of every compact operator is mostly countable, we conclude that the operators $U, V \in B(X)$ for each $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $p \geq 1$, are not compact. Also $W_1, W_2 \in B(X)$ for each $X \in \{c_0(\mathbb{Z}), \ell_p(\mathbb{Z})\}$, $p \geq 1$, are not compact.

For more about shifts we refer the reader to [1] and [2].

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